ON THE TANGENT CONES TO PLURISUBHARMONIC CURRENTS

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Abstract. In this paper, we study the existence of the tangent cone to a positive plurisubharmonic or plurisuperharmonic current with a suitable condition. Some Estimates of the growth of the Lelong functions associated to the current and to its dd^c are given to ensure the existence of the blow-up of this current. A second proof for the existence of the tangent cone is derived from these estimates.

Sur les cônes tangents au courants plurisousharmoniques.
Résumé. Dans cet article, nous étudions l'existence du cône tangent à un courant positif plurisousharmonique ou plurisurharmonique avec une condition convenable. Des estimations de croissance des fonctions de Lelong associées au courant et à son dd^c sont données pour assurer l'existence du relèvement de ce courant. Une deuxième preuve de l'existence du cône tangent se déduit de ces estimations.

1. Introduction

Let T be a positive current of bidimension (p,p) on a neighborhood Ω of 0 in $\mathbb{C}^n$, $0<p<n$, and h_a be the complex dilatation on $\mathbb{C}^n$ ($h_a(z)=az$) with $a\in\mathbb{C}^*$. In this paper we study the existence of the weak limit of the family of currents $(h_a^\star T)_a$ when $|a|$ tends to 0. A such limit is called tangent cone to T. The case of analytic sets was studied by Thie in 1967 and then by King in 1971. Thus they prove that the tangent cone to the current $[A]$ (current of integration over the analytic set A) is given by the current of integration over the geometric tangent cone to the analytic set A. However, this statement is not true in case of positive closed currents and a counterexample was given by Kiselman where he constructed a psh function u such that the current dd^cu doesn't have a tangent cone. For this reason, to ensure the existence of the tangent cone we need some conditions. For closed positive currents, Blel, Demailly and Mouzali gave two independent conditions, where each one ensures the existence of the tangent cone. In this paper, we show that the second condition, condition (b) in [1], is even

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sufficient in the case of positive plurisubharmonic ($dd^cT \geq 0$) or plurisuperharmonic ($dd^cT \leq 0$) currents. Precisely we have

Theorem 1. ***(Main result)*** *Let T be a positive plurisubharmonic (resp. plurisuperharmonic) current of bidimension (p,p) on Ω, $0 < p < n$. Then the tangent cone to T at 0 exists if, for $r_0 > 0$, we have*

$$\int_0^{r_0} \frac{\nu_T(r) - \nu_T(0)}{r} dr < +\infty$$

resp.

$$\int_0^{r_0} \frac{\nu_{dd^cT}(t)}{t} dt > -\infty \quad \textit{and} \quad \int_0^{r_0} \frac{|\nu_T(r) - \nu_T(0)|}{r} dr < +\infty.$$

We start our paper by giving some preliminary results. Next, we give a direct proof of the main result. Finely, we study the problem of restriction of positive currents along analytic sets and we conclude a second (partial) proof of the main result.

1.1. **Lelong numbers.** Let now recall some notations and preliminary results useful in the following.
For every $r > 0$, $r_2 > r_1 > 0$ and $z_0 \in \mathbb{C}^n$, we set

$$\begin{aligned}
&B(z_0, r) := \{z \in \mathbb{C}^n;\ |z - z_0| < r\}\\
&B(z_0, r_1, r_2) := \{z \in \mathbb{C}^n;\ r_1 \leq |z - z_0| < r_2\} = B(z_0, r_2) \smallsetminus B(z_0, r_1)\\
&\beta_{z_0} := dd^c|z - z_0|^2 = \frac{i}{2\pi}\partial\overline{\partial}|z - z_0|^2, \quad \alpha_{z_0} := dd^c \log|z - z_0|^2.
\end{aligned}$$

When $z_0 = 0$, we omit z_0 in previous notations and we use only $B(r)$, $B(r_1, r_2)$, β and α instead of $B(0, r)$, $B(0, r_1, r_2)$, β_0 and α_0 respectively.

Let T be a positive plurisubharmonic or plurisuperharmonic current of bidimension (p,p) on Ω and $z_0 \in \Omega$. Let $R > 0$ such that $B(z_0, R) \subset\subset \Omega$. For all $0 < r < R$, we set $\nu_T(z_0, r) = \frac{1}{r^{2p}} \int_{B(z_0,r)} T \wedge \beta_{z_0}^p$ the projective mass of T. The well-known following lemma will be used frequently in the hole of this paper.

Lemma 1. *(Lelong-Jensen formula) Let S be a positive plurisubharmonic or plurisuperharmonic current of bidimension (p,p) on Ω and $z_0 \in \Omega$. Then,*

for all $0 < r_1 < r_2 < R$,
(1.1)

$$\begin{aligned}
\nu_S(z_0,r_2)-\nu_S(z_0,r_1) &= \frac{1}{r_2^{2p}}\int_{B(z_0,r_2)} S\wedge\beta_{z_0}^p - \frac{1}{r_1^{2p}}\int_{B(z_0,r_1)} S\wedge\beta_{z_0}^p \\
&= \int_{r_1}^{r_2}\left(\frac{1}{t^{2p}}-\frac{1}{r_2^{2p}}\right)tdt\int_{B(z_0,t)} dd^cS\wedge\beta_{z_0}^{p-1} \\
&\quad +\left(\frac{1}{r_1^{2p}}-\frac{1}{r_2^{2p}}\right)\int_0^{r_1} tdt\int_{B(z_0,t)} dd^cS\wedge\beta_{z_0}^{p-1} \\
&\quad +\int_{B(z_0,r_1,r_2)} S\wedge\alpha_{z_0}^p.
\end{aligned}$$

According to Lemma 1, if T is positive plurisubharmonic then $\nu_T(z_0,.)$ is a non-negative increasing function on $]0,R[$, so the Lelong number $\nu_T(z_0) := \lim_{r\to 0^+}\nu_T(z_0,r)$ of T at z_0 exists.
For positive plurisuperharmonic currents, the existence of Lelong numbers was treated by the first author and he proved that Lelong numbers do not depend on the system of coordinates. We cite the main result of [4].

Theorem 2. *Let T be a positive plurisuperharmonic current of bidimension (p,p) on Ω, $0<p<n$, and $z_0\in\Omega$. We assume that T satisfies condition $(C)_{z_0}$ given by:*

$$(C)_{z_0}: \qquad \int_0^{r_0}\frac{\nu_{dd^cT}(z_0,t)}{t}dt > -\infty$$

for some $0<r_0\le R$. Then, the Lelong number $\nu_T(z_0)$ of T at z_0 exists.

Proof. For every $0<r<R$, we set

$$\Lambda_{z_0}(r)=\nu_T(z_0,r)+\int_0^r\left(\frac{t^{2p}}{r^{2p}}-1\right)\frac{\nu_{dd^cT}(z_0,t)}{t}dt.$$

Thanks to condition $(C)_{z_0}$ and using the fact that $\nu_{dd^cT}(z_0,.)$ is non-positive on $]0,R[$, one can deduce that Λ_{z_0} is well defined and non-negative on $]0,R[$.
For $0<r_1<r_2<R$, Lemma 1 gives

$$\begin{aligned}
\Lambda_{z_0}(r_2)-\Lambda_{z_0}(r_1) &= \nu_T(z_0,r_2)-\nu_T(z_0,r_1)+\frac{1}{r_2^{2p}}\int_0^{r_2} t^{2p-1}\nu_{dd^cT}(z_0,t)dt \\
&\quad -\frac{1}{r_1^{2p}}\int_0^{r_1} t^{2p-1}\nu_{dd^cT}(z_0,t)dt-\int_{r_1}^{r_2}\frac{\nu_{dd^cT}(z_0,t)}{t}dt \\
&= \int_{B(z_0,r_1,r_2)} T\wedge\alpha_{z_0}^p \ge 0.
\end{aligned}$$

Therefore, Λ_{z_0} is a non-negative increasing function on $]0,R[$, and this implies the existence of the limit $\varrho := \lim_{r\to 0^+}\Lambda_{z_0}(r)$. The hypothesis of integrability of $\nu_{dd^cT}(z_0,t)/t$ and the fact that (t^p/r^p-1) is uniformly bounded give

$$\lim_{r\to 0^+}\int_0^r\left(\frac{t^{2p}}{r^{2p}}-1\right)\frac{\nu_{dd^cT}(z_0,t)}{t}dt=0.$$

Hence, $\varrho = \lim_{r\to 0^+} \Lambda_{z_0}(r) = \lim_{r\to 0^+} \nu_T(z_0, r) = \nu_T(z_0)$. $\square$

The following example proves that Condition (C) is not necessary in Theorem 2 for the existence of Lelong number.

Example 1. Let $T_0 = du \wedge d^c u$ where $u(z) = \log |z|^2$. Then T_0 is a positive current of bidimension $(1, 1)$ on $\mathbb{C}^2$. Furthermore one has $dd^c T_0 = -(dd^c u)^2 = -\delta_0$ (Dirac) is negative on $\mathbb{C}^2$ and $\nu_{dd^c T_0}(0) = -1$, so Condition $(C)_0$ is not satisfied. In the other hand, a simple computation shows that

$$\nu_{T_0}(r) = \frac{1}{4\pi^2 r^2} \int_{|z|<r} \frac{1}{|z|^2} i dz_1 \wedge d\overline{z}_1 \wedge i dz_2 \wedge d\overline{z}_2 = \frac{1}{4}.$$

An open problem arises from this part which is to study the set $\mathscr{E}_\infty(T)$ of points z in Ω for which the Lelong number of T at z doesn't exist. If T is positive plurisubharmonic then $\mathscr{E}_\infty(T)$ is empty, but if T is positive plurisuperharmonic then $\mathscr{E}_\infty(T)$ can be non-empty, for example $\mathscr{E}_\infty(T_1) = \{0\}$ where $T_1 = -\log |z_1|^2 [z_2 = 0]$ on $\mathbb{C}^2$.
We remark that if T is positive plurisuperharmonic, then $\mathscr{E}_\infty(T) \subset \mathscr{F}_\infty(T)$ where

$$\begin{aligned} \mathscr{F}_\infty(T) &:= \left\{ z \in \Omega;\ \frac{\nu_{dd^c T}(z, t)}{t} \notin L^1(\vartheta(0)) \right\} \\ &= \{z \in \Omega;\ \nu_{dd^c T}(z) < 0\} \cup \{z \in \mathscr{F}_\infty(T);\ \nu_{dd^c T}(z) = 0\} \\ &=: \mathscr{F}^1_\infty(T) \cup \mathscr{F}^2_\infty(T). \end{aligned}$$

The subset $\mathscr{F}^1_\infty(T)$ is pluripolar in Ω. Indeed, thanks to Siu's theorem,

$$\mathscr{F}^1_\infty(T) = \bigcup_{j\in\mathbb{N}^*} \left\{ z \in \Omega;\ \nu_{dd^c T}(z) \le -\frac{1}{j} \right\}$$

is a countable union of analytic sets, because $dd^c T$ is a negative closed current. For the second subset $\mathscr{F}^2_\infty(T)$, we conjecture that it is also pluripolar.

1.2. **A Structure theorem.** In this part, we study a geometric structure of the support of a positive plurisuperharmonic current in Ω. All results given are available in any complex manifold of dimension n.
Our aim here is to prove the following theorem:

Theorem 3. *Let T be a positive plurisuperharmonic current of bidimension (p, p) on an open set Ω of $\mathbb{C}^n$ $(1 \le p \le n - 1)$ such that the function $t \mapsto \frac{\nu_{dd^c T}(z,t)}{t}$ is locally uniformly integrable in neighborhood of points of $X = Supp(T)$. Assume that there exists a real number $\delta > 0$ such that the level-set $E_\delta := \{z \in \Omega;\ \nu_T(z) \ge \delta\}$ is dense in X. Then X is a complex subvariety of pure dimension p of Ω and there exists a weakly plurisubharmonic negative function φ on X such that $T = -\varphi[X]$.*

A similar result was given by Dinh-Lawrence [2] in the case of positive plurisubharmonic currents.
To prove this theorem we have to recall some results.

Definition 1. Let Z be a closed subset of Ω and p an integer, $0 < p < n$ ($n \geq 2$). We say that Z is $p-$pseudoconcave in Ω if for every open set $\mathcal{U} \subset\subset \Omega$ and every holomorphic map f from a neighborhood of $\overline{\mathcal{U}}$ into $\mathbb{C}^p$ we have $f(Z \cap \mathcal{U}) \subset \mathbb{C}^p \smallsetminus V$ where V is the bounded component of $\mathbb{C}^p \smallsetminus f(Z \cap \partial\mathcal{U})$.

Pseudoconcave sets were studied by Fornæss-Sibony [3], Dinh-Lawrence [2] and others. It was shown that the support of a positive plurisuperharmonic current is an example of pseudoconcave sets, Precisely we have:

Lemma 2. *(See* [2, 3]*) Let T be a positive plurisuperharmonic current of bidimension (p,p) on an open set Ω of $\mathbb{C}^n$ ($1 \leq p \leq n-1$). Then $X := Supp(T)$ is $p-$pseudoconcave in Ω.*

The fundamental tool in the proof of Theorem 3 is the following lemma.

Lemma 3. *(See* [2]*) Let Ω be a complex manifold of dimension $n \geq 2$ and X a $p-$pseudoconcave subset of Ω. Let K be a compact subset of Ω which admits a Stein neighborhood. Assume that the $2p-$dimensional Hausdorff measure of $X \smallsetminus K$ is locally finite in $\Omega \smallsetminus K$. Then X is a complex subvariety of pure dimension p of Ω.*

Now we can prove Theorem 3.

Proof. It is sufficient to prove that X is a complex subvariety of pure dimension p of Ω. Thanks to lemmas 2 and 3, we have to prove that X has locally finite $\mathscr{H}^{2p}$ Hausdorff measure. In fact we prove that $X = E_\delta$ and E_δ has locally finite $\mathscr{H}^{2p}$ measure. For the last affirmation, we remark that for every $z \in E_\delta$ one has

$$\delta \leq \nu_T(z) = \lim_{t\to 0} \frac{\sigma_T(B(z,t))}{t^{2p}}$$

which proves that E_δ has locally finite $\mathscr{H}^{2p}$ measure. Since $E_\delta \subset X$, To prove $X = E_\delta$, it suffice to prove E_δ is closed in Ω. For this, Let $(\xi_j)_j \subset E_\delta$ and $\xi_j \underset{j\to+\infty}{\longrightarrow} \xi \in \Omega$. Fix an $r_0 > 0$ such that $B(\xi, r_0) \subset \Omega$ and j_0 such that $\xi_j \in B(\xi, r_0)$ for all $j \geq j_0$. Since $\sigma_T := T \wedge \beta^p$ is a positive measure, then for every $0 < r < r_0$ and for every $j \geq j_1$ (for which $0 < |\xi - \xi_j| < r$), we have $\sigma_T(B(\xi, r)) \geq \sigma_T(B(\xi_j, r - |\xi - \xi_j|))$. One has

$$\delta \leq \nu_T(\xi_j) \leq \Lambda_{\xi_j}(s_j) := \nu_T(\xi_j, s_j) + \int_0^{s_j} \left(\frac{t^{2p}}{s_j^{2p}} - 1\right) \frac{\nu_{dd^cT}(\xi_j, t)}{t} dt.$$

where $s_j = r - |\xi - \xi_j|$. So,

$$\begin{aligned}\sigma_T(B(\xi, r)) &\geq \delta s_j^{2p} - s_j^{2p} \int_0^{s_j} \left(\frac{t^{2p}}{s_j^{2p}} - 1\right) \frac{\nu_{dd^cT}(\xi_j, t)}{t} dt \\ &\geq \delta s_j^{2p} - s_j^{2p} \mathscr{I}_j.\end{aligned}$$

We set $\mathscr{I}_j = \mathscr{I}_{j,1} + \mathscr{I}_{j,2}$ where

$$\begin{aligned} 0 \le \mathscr{I}_{j,1} &= \int_0^{|\xi-\xi_j|} \left(\frac{t^{2p}}{(r-|\xi-\xi_j|)^{2p}} - 1 \right) \frac{\nu_{dd^cT}(\xi_j,t)}{t} dt \\ &\le -\int_0^{|\xi-\xi_j|} \frac{\nu_{dd^cT}(\xi_j,t)}{t} dt \underset{\xi_j \to \xi}{\longrightarrow} 0. \end{aligned}$$

and

$$\begin{aligned} \mathscr{I}_{j,2} &= \int_{|\xi-\xi_j|}^{r-|\xi-\xi_j|} \left(\frac{t^{2p}}{(r-|\xi-\xi_j|)^{2p}} - 1 \right) \frac{\nu_{dd^cT}(\xi_j,t)}{t} dt \\ &\le \int_{|\xi-\xi_j|}^{r-|\xi-\xi_j|} \left(\frac{t^{2p}}{(r-|\xi-\xi_j|)^{2p}} - 1 \right) \frac{\sigma_{dd^cT}(B(\xi, t+|\xi-\xi_j|))}{t^{2p-1}} dt \\ &\underset{\xi_j \to \xi}{\longrightarrow} \int_0^r \left(\frac{t^{2p}}{r^{2p}} - 1 \right) \frac{\sigma_{dd^cT}(B(\xi,t))}{t^{2p-1}} dt = \int_0^r \left(\frac{t^{2p}}{r^{2p}} - 1 \right) \frac{\nu_{dd^cT}(\xi,t)}{t} dt. \end{aligned}$$

Hence

$$\nu_T(\xi, r) \ge \delta - \int_0^r \left(\frac{t^{2p}}{r^{2p}} - 1 \right) \frac{\nu_{dd^cT}(\xi,t)}{t} dt.$$

If $r \to 0$, we obtain $\nu_T(\xi) \ge \delta$; so $\xi \in E_\delta$ which proves that E_δ is closed. We can conclude from previous computations the upper-semi-continuity of ν_T on Ω. $\square$

2. Proof of the main result

In this part, we study the existence of the tangent cone to positive plurisubharmonic or plurisuperharmonic currents on an open neighborhood Ω of 0 in $\mathbb{C}^n$. The principal result will be proved partially with a different way in the third section.

In the following, we will use $\mathscr{P}_p^+(\Omega)$ (resp. $\mathscr{P}_p^-(\Omega)$) to indicate the set of positive plurisubharmonic currents (resp. positive plurisuperharmonic currents satisfying condition $(C)_0$) of bidimension (p,p) on Ω where $0 < p < n$.

Theorem 1. **(Main result)** *Let $T \in \mathscr{P}_p^\pm(\Omega)$. Then the tangent cone to T exists when*

$$\int_0^{r_0} \frac{|\nu_T(r) - \nu_T(0)|}{r} dr < +\infty$$

where r_0 is a positive real such that $B(r_0) \subset\subset \Omega$.

This theorem is due to Blel-Demailly-Mouzali in case of positive closed currents. In [6], Haggui proved the same result for $T \in \mathscr{P}_p^+(\Omega)$. His proof is based on the potential current associated to dd^cT. Here, we present a proof which is different from Haggui's one.

Remark 1. Condition $(C)_0$ is not necessary in Theorem 1. In fact the current T_0 of Example 1 admits a tangent cone and doesn't satisfy Condition $(C)_0$. Indeed $h_a^\star T_0 = T_0$ (T_0 is conic on $\mathbb{C}^2$).

Proof. Let $T \in \mathscr{P}_p^+(\Omega)$ (resp. $T \in \mathscr{P}_p^-(\Omega)$). Using $h_a^\star T$ in equality (1.1) and the equality $\nu_{h_a^\star T}(r) = \nu_T(|a|r)$ for all $|a| < r_0/r$, we find

$$\int_{B(r)} h_a^\star T \wedge \beta^p \leq \nu_T(r_0) r^{2p}, \quad \forall\, |a| \leq \frac{r_0}{r}. \tag{2.1}$$

resp.

$$\int_{B(r)} h_a^\star T \wedge \beta^p \leq \Lambda_0(r_0) r^{2p}, \quad \forall\, |a| \leq \frac{r_0}{r}. \tag{2.2}$$

In both cases, equations (2.1) and (2.2) give the mass of $(h_a^\star T)$ is uniformly small in the neighborhood of 0. Hence $(h_a^\star T)$ converges weakly on $\mathbb{C}^n$ if and only if it converges weakly in the neighborhood of every point $z^0 \in \mathbb{C}^n \smallsetminus \{0\}$. After a suitable dilatation and a unitary changement of coordinates, we can assume that $z^0 = (0, ..., 0, z_n^0)$ where $1/2 < z_n^0 < 1$. We use projective coordinates and we set

$$w_1 = \frac{z_1}{z_n}, ...,\ w_{n-1} = \frac{z_{n-1}}{z_n},\ w_n = z_n$$

and

$$T = 2^{-q} i^{q^2} \sum_{|I|=|J|=q} T_{I,J} dw_I \wedge d\overline{w}_J$$

where $q = n - p$. The dilatation h_a is written as $h_a:\ w = (w', w_n) \mapsto (w', aw_n)$ with $w' = (w_1, ..., w_{n-1})$. We verify that the coefficients $T_{I,J}^a$ of $h_a^\star T$ are given by

$$T_{I,J}^a(w) = \begin{cases} T_{I,J}(w', aw_n) & if \quad n \notin I,\ n \notin J \\ a\, T_{I,J}(w', aw_n) & if \quad n \in I,\ n \notin J \\ \overline{a}\, T_{I,J}(w', aw_n) & if \quad n \notin I,\ n \in J \\ |a|^2\, T_{I,J}(w', aw_n) & if \quad n \in I,\ n \in J \end{cases} \tag{2.3}$$

The proof of the main result when $T \in \mathscr{P}_p^-(\Omega)$ is similar to the case $T \in \mathscr{P}_p^+(\Omega)$ with some simple modification, for this reason we will continuous this proof with $T \in \mathscr{P}_p^+(\Omega)$.
We need the following lemma:

Lemma 4. *Let U be the neighborhood of z^0 given by*

$$U = \{z \in \mathbb{C}^n;\ |z| < 1,\ 1/2 < |z_n| < 1\} \subset B(1/2, 1).$$

We consider the two functions $\gamma_T(r) = \nu_T(r) - \nu_T(r/2)$ and $\gamma_{dd^cT}(r) = \nu_{dd^cT}(r) - \nu_{dd^cT}(r/2)$ defined on $]0, R[$. For $r_0 < R$, there exist three positive constants $C_1,\ C_2$ and $C_3 > 0$ such that for $|a| < r_0$, the measure $T_{I,J}^a$ satisfies the following estimates

$$\int_U |T_{I,J}^a| \leq \begin{cases} C_1 & \textit{for all } I,\ J \\ C_2\left(\gamma_T(|a|) + \gamma_{dd^cT}(|a|)\right) & \textit{if } n \in I, \textit{ and } n \in J \\ C_3\sqrt{\gamma_T(|a|) + \gamma_{dd^cT}(|a|)} & \textit{if } n \in I, \textit{ or } n \in J \end{cases} \tag{2.4}$$

The proof of this lemma will be done later, so we can now continuous our proof.
Thanks to Lemma 4, $T^a_{I,J}$ tends to 0 in mass for I or J containing n, hence, to finish the proof it suffice to study the weak convergence of measures $T^a_{I,J}$ when $n \notin I$ and $n \notin J$.
Let $\varphi \in \mathscr{D}(U)$. For $n \notin I = \{i_1, ..., i_q\},\ n \notin J = \{j_1, ..., j_q\}$, we set

$$f_{I,J}(a) = \int_U T^a_{I,J}(w)\varphi(w)d\tau(w) = \int_U T_{I,J}(w', aw_n)\varphi(w)d\tau(w).$$

$f_{I,J}$ is $\mathcal{C}^\infty$ on $D^*(0,R) := \{a \in \mathbb{C};\ 0 < |a| < R\}$ and it is bounded in a neighborhood of 0. The problem is to show that $f_{I,J}(a)$ admits a limit when $a \to 0$. The idea is to estimate $\Delta f_{I,J}$ in a neighborhood of 0. We have

$$\frac{\partial^2 f_{I,J}}{\partial a \partial \overline{a}}(a) = \int_U |w_n|^2 \frac{\partial^2 T_{I,J}}{\partial w_n \partial \overline{w}_n}(w', aw_n)\varphi(w)d\tau(w).$$

We remark that the coefficient of $dw_{I\cup\{n\}} \wedge d\overline{w}_{J\cup\{n\}}$ in the expression of dd^cT is

$$\begin{aligned}(dd^cT)_{I\cup\{n\},J\cup\{n\}} \quad = \quad & (-1)^q \frac{\partial^2 T_{I,J}}{\partial w_n \partial \overline{w}_n} + \sum_{k,s=1}^{q} (-1)^{k+q+s-2} \frac{\partial^2 T_{I(k),J(s)}}{\partial w_{i_k} \partial \overline{w}_{j_s}} \\ & + \sum_{s=1}^{q} (-1)^{s-1} \frac{\partial^2 T_{I,J(s)}}{\partial w_n \partial \overline{w}_{j_s}} + \sum_{k=1}^{q} (-1)^{k-1} \frac{\partial^2 T_{I(k),J}}{\partial w_{i_k} \partial \overline{w}_n}\end{aligned}$$

where $I(k) = I \smallsetminus \{i_k\} \cup \{n\}$ and $J(s) = J \smallsetminus \{j_s\} \cup \{n\}$. It follows from equality(2.3), that

$$\begin{aligned}\frac{\partial^2 f_{I,J}}{\partial a \partial \overline{a}}(a) \quad = \quad & (-1)^q \int_U \frac{|w_n|^2}{|a|^2} (dd^cT)^a_{I\cup\{n\},J\cup\{n\}} \varphi(w)d\tau(w) \\ & + \sum_{k,s=1}^{q} (-1)^{k+s-1} \int_U \frac{1}{|a|^2} T^a_{I(k),J(s)} \frac{\partial^2 \varphi}{\partial w_{i_k} \partial \overline{w}_{j_s}} d\tau(w) \\ & + \sum_{k=1}^{q} (-1)^{q+k} \int_U \frac{1}{a} T^a_{I(k),J} \frac{\partial^2 \varphi}{\partial w_{i_k} \partial \overline{w}_n} d\tau(w) \\ & + \sum_{s=1}^{q} (-1)^{q+s} \int_U \frac{1}{\overline{a}} T^a_{I,J(s)} \frac{\partial^2 \varphi}{\partial w_n \partial \overline{w}_{j_s}} d\tau(w).\end{aligned}$$

Thanks to lemma 4, one has

$$\begin{aligned}\left|\frac{\partial^2 f_{I,J}}{\partial a \partial \overline{a}}(a)\right| \quad \leq \quad & C_1 \frac{\gamma_{dd^cT}(|a|)}{|a|^2} + C_2 \frac{\gamma_T(|a|) + \gamma_{dd^cT}(|a|)}{|a|^2} \\ & \qquad + C_3 \frac{\sqrt{\gamma_T(|a|) + \gamma_{dd^cT}(|a|)}}{|a|} \\ \leq \quad & C \left(\tfrac{\gamma_T(|a|) + \gamma_{dd^cT}(|a|)}{|a|^2} + \tfrac{\sqrt{\gamma_T(|a|) + \gamma_{dd^cT}(|a|)}}{|a|} \right) = C\psi(|a|).\end{aligned}$$

Thanks to [1, lemme 3.6], $f_{I,J}(a)$ admits a limit at 0 if ψ satisfies

$$\int_0^{r_0} r|\log r|\psi(r)dr < +\infty.$$

A simple computation (see [1]) shows that

$$\int_0^{r_0} \frac{\gamma_T(r) + \gamma_{dd^cT}(r)}{r}|\log r|dr < +\infty$$

is equivalent to

$$\int_0^{r_0} \frac{\nu_T(r) - \nu_T(0)}{r}dr < +\infty \text{ and } \int_0^{r_0} \frac{\nu_{dd^cT}(r)}{r}dr < +\infty.$$

those conditions are cited in the hypothesis of the main result. Hence, we have

$$\int_0^{r_0} \frac{\gamma_T(r) + \gamma_{dd^cT}(r)}{r}|\log r|dr < +\infty. \tag{2.5}$$

Thanks to Cauchy-Schwarz inequality, (2.5) gives

$$\begin{aligned}\int_0^{r_0} \sqrt{\gamma_T(r) + \gamma_{dd^cT}(r)}|\log r|dr \quad &\leq \quad \left(\int_0^{r_0} \frac{\gamma_T(r) + \gamma_{dd^cT}(r)}{r}|\log r|dr\right)^{1/2} \\ &\qquad\qquad \times \left(\int_0^{r_0} r|\log r|dr\right)^{1/2} \\ &< \quad +\infty.\end{aligned}$$

Therefore,

$$\int_0^{r_0} r|\log r|\psi(r)dr < +\infty$$

which completes the proof of Theorem 1. □

To prove Lemma 4, we need Demailly's inequality: *If*

$$S = 2^{-q} i^{q^2} \sum_{|I|=|J|=q} S_{I,J} dw_I \wedge d\overline{w}_J$$

is a positive $(q,q)-$*current then for all* $(\lambda_1, ..., \lambda_n) \in]0, +\infty[^n$ *we have*

$$\lambda_I \lambda_J |S_{I,J}| \leq 2^q \sum_{M \in \mathscr{M}_{I,J}} \lambda_M S_{M,M} \tag{2.6}$$

where $\lambda_I = \lambda_{i_1} ... \lambda_{i_q}$ *if* $I = (i_1, ..., i_q)$ *and the sum is taken over the set of* $q-$*index* $\mathscr{M}_{I,J} = \{M;\ |M| = q,\ I \cap J \subset M \subset I \cup J\}$.

Now, we can prove Lemma 4.

Proof.

- The set $\overline{U}$ is compact and the $(1,1)-$forme β is smooth and positive, so we have $\beta \geq C_4 dd^c|w|^2$ on U. Inequality (2.1), with $r=1$, implies $\int_U T^a_{I,I} \leq C_5$ uniformly to a for $|a|<r_0$. Demailly's inequality (2.6), with the choice $\lambda_1 = ... = \lambda_n = 1$, gives
$$\int_U |T^a_{I,J}| \leq C_6 \sum_{M\in \mathscr{M}_{I,J}} \int_U T^a_{M,M} \leq C_1$$
so the first estimate in (2.4) is proved.
- To prove the second estimate, we remark that we have $\alpha \geq C_7\beta'$ on U where $\beta' = dd^c|w'|^2$. Indeed, $\alpha = dd^c\log(1+|w'|^2) \geq \frac{1}{(1+|w'|^2)^2}\beta' \geq \frac{1}{4}\beta'$ on U. Hence
$$\begin{aligned}\int_U \sum_{I\ni n} T^a_{I,I} &= \int_U h_a^\star T\wedge (dd^c|w'|^2)^p \\ &\leq C_8 \int_U h_a^\star T\wedge \alpha^p \leq C_8 \int_{B(1/2,1)} h_a^\star T\wedge\alpha^p.\end{aligned}$$
Thanks to Lelong-Jensen formula, with $r_2=1$ and $r_1=1/2$, one has
$$\begin{aligned}\int_U \sum_{I\ni n} T^a_{I,I} &\leq C_8 \int_{B(1/2,1)} h_a^\star T\wedge\alpha^p \\ &\leq C_8\Bigg[\nu_T(|a|)-\nu_T(|a|/2) - \int_{\frac{1}{2}}^1 \left(\frac{1}{t^{2p}}-1\right) t^{2p-1}\nu_{dd^c(h_a^\star T)}(t)dt \\ &\qquad - \left(\frac{1}{2^{2p}}-1\right)\int_0^{\frac{1}{2}} t^{2p-1}\nu_{dd^c(h_a^\star T)}(t)dt\Bigg] \\ &\leq C_8\Bigg[\nu_T(|a|)-\nu_T(|a|/2) - \int_{\frac{1}{2}}^1 \frac{\nu_{dd^cT}(|a|t)}{t}dt \\ &\qquad + \int_0^1 t^{2p-1}\nu_{dd^cT}(|a|t)dt\Bigg] \\ &\leq C_8(\nu_T(|a|)-\nu_T(|a|/2)) + C_9(\nu_{dd^cT}(|a|)-\nu_{dd^cT}(|a|/2)) \\ &\leq C_8\gamma_T(|a|) + C_9\gamma_{dd^cT}(|a|)\end{aligned}$$
because ν_{dd^cT} is a non-negative increasing function. The second estimate is proved for $I=J\ni n$.
for the general case, $I,J\ni n$, we use Demailly's inequality (2.6) with $\lambda_1 = ... = \lambda_n = 1$, to obtain
$$\int_U |T^a_{I,J}| \leq C_{10} \sum_{M\in\mathscr{M}_{I,J}} \int_U T^a_{M,M} \leq C_2(\gamma_T(|a|)+\gamma_{dd^cT}(|a|))$$
and the second estimate is proved.
- For the third estimate, it suffice to assume that $n\in I$ and $n\notin J$. Again thanks to Demailly's inequality (2.6), with $\lambda_1 = ... = \lambda_{n-1} =$

1 and $\lambda_n > 0$, we have

$$\begin{aligned}\lambda_n \int_U |T^a_{I,J}| &\leq C_{11} \int_U \left(\sum_{n\notin M\in \mathscr{M}_{I,J}} T^a_{M,M} + \lambda_n^2 \sum_{n\in M\in \mathscr{M}_{I,J}} T^a_{M,M} \right) \\ &\leq C_{12} + C_{13}\lambda_n^2(\gamma_T(|a|) + \gamma_{dd^cT}(|a|)).\end{aligned}$$

The third estimate can be deduced from the choice

$$\lambda_n = \frac{1}{\sqrt{\gamma_T(|a|) + \gamma_{dd^cT}(|a|)}}.$$

□

3. Pull-Back of positive currents

Let $\mathbb{C}^n[0] := \{(z, L) \in \mathbb{C}^n \times \mathbb{P}^{n-1};\ z \in L\}$ and T be a positive current on $\mathbb{C}^n$. In this section, We study the existence of a positive current $\widehat{T}$ on $\mathbb{C}^n[0]$ such that $\pi_\star\widehat{T} = T$ where $\pi : \mathbb{C}^n[0] \longrightarrow \mathbb{C}^n$ is the canonical projection; in this statement, we say that T admits a blow-up by π over 0. We give a positive answer in case of positive plurisubharmonic or plurisuperharmonic currents. Finely, we apply this result to give a second proof of the main result with a supplementary condition in case of positive plurisuperharmonic currents.

Proposition 1. *Let $T \in \mathscr{P}^+_p(\mathbb{C}^n)$ (resp. $T \in \mathscr{P}^-_p(\mathbb{C}^n)$). Then, T admits a blow-up $\widehat{T}$ (a positive current on $\mathbb{C}^n[0]$ such that $\pi_\star\widehat{T} = T$). Furthermore, for $r > 0$ one has*

$$||\widehat{T}||(\pi^{-1}(B(r))) \leq \nu_T(r) - \nu_T(0) + C_r\nu_T(r). \tag{3.1}$$

resp.

$$\begin{aligned}||\widehat{T}||(\pi^{-1}(B(r))) \;&\leq\; |\nu_T(r) - \nu_T(0)| + C_r\nu_T(r) - C'_r\nu_{dd^cT}(r) \\ &\qquad + \int_0^r \left(\frac{t^{2p}}{r^{2p}} - 1\right) \frac{\nu_{dd^cT}(t)}{t} dt\end{aligned} \tag{3.2}$$

where $C_r := \sum_{k=1}^p C^k_p r^{2k}$ and $C'_r := \sum_{k=1}^p \frac{C^k_p}{2k} r^{2k}$.

Proposition 1 is proved by Giret [5] in the case of positive closed currents.

Proof. Let $T \in \mathscr{P}^\pm_p(\mathbb{C}^n)$. The canonical projection π is a submersion from $\mathbb{C}^n[0] \smallsetminus \pi^{-1}(\{0\})$ to $\mathbb{C}^n \smallsetminus \{0\}$, so $\mathcal{T} := \pi^\star(T_{|\mathbb{C}^n\smallsetminus\{0\}})$ exists. Furthermore $\mathcal{T}$ has a locally finite mass near every point of $\pi^{-1}(\{0\})$. Let $\widehat{T}$ be the trivial extension of $\mathcal{T}$ by zero over $\mathbb{P}^{n-1} \approx \{0\} \times \mathbb{P}^{n-1} = \pi^{-1}(\{0\})$, $\widehat{T}$ is positive on $\mathbb{C}^n[0]$ and $\pi_\star\widehat{T} = T$. To prove Inequality (3.1) (resp. (3.2)), let ω be the

Kähler form of $\mathbb{C}^n[0]$. Then, for $0<\epsilon<r$, we have

$$\begin{aligned}||\mathcal{T}||(\pi^{-1}(B(\epsilon,r))) &= \int_{B(\epsilon,r)} T\wedge\pi_\star\omega^p = \int_{B(\epsilon,r)} T\wedge(\alpha+\beta)^p\\ &= \sum_{k=0}^{p} C_p^k \int_{B(\epsilon,r)} T\wedge\alpha^{p-k}\wedge\beta^k\\ &= \sum_{k=0}^{p-1} C_p^k \int_{B(\epsilon,r)} T\wedge\beta^k\wedge\alpha^{p-k} + \int_{B(\epsilon,r)} T\wedge\beta^p.\end{aligned}$$

- *First case: $T\in\mathscr{P}_p^+(\mathbb{C}^n)$.* For every $0\le k\le p-1$, Lelong-Jensen formula applied to the current $T\wedge\beta^k$ gives

$$\begin{aligned}\int_{B(\epsilon,r)} T\wedge\beta^k\wedge\alpha^{p-k} &= \frac{1}{r^{2(p-k)}}\int_{B(r)} T\wedge\beta^p - \frac{1}{\epsilon^{2(p-k)}}\int_{B(\epsilon)} T\wedge\beta^p\\ &\quad -\int_\epsilon^r\left(\frac{1}{t^{2(p-k)}}-\frac{1}{r^{2(p-k)}}\right)tdt\int_{B(t)} dd^cT\wedge\beta^{p-1}\\ &\quad -\left(\frac{1}{\epsilon^{2(p-k)}}-\frac{1}{r^{2(p-k)}}\right)\int_0^\epsilon tdt\int_{B(t)} dd^cT\wedge\beta^{p-1}\\ &\le r^{2k}\nu_T(r)-\epsilon^{2k}\nu_T(\epsilon)\end{aligned}$$

  hence,

$$\begin{aligned}||\mathcal{T}||(\pi^{-1}(B(\epsilon,r))) &= r^{2p}\nu_T(r)-\epsilon^{2p}\nu_T(\epsilon)+\sum_{k=0}^{p-1} C_p^k\int_{B(\epsilon,r)} T\wedge\beta^k\wedge\alpha^{p-k}\\ &\le \sum_{k=0}^{p} C_p^k\left(r^{2k}\nu_T(r)-\epsilon^{2k}\nu_T(\epsilon)\right)\\ &\le \nu_T(r)-\nu_T(\epsilon)+\sum_{k=1}^{p} C_p^k\left(r^{2k}\nu_T(r)-\epsilon^{2k}\nu_T(\epsilon)\right).\end{aligned}$$

  which is bounded independently to ϵ. Inequality (3.1) is obtained by tending ϵ to 0.
- *Second case: $T\in\mathscr{P}_p^-(\mathbb{C}^n)$.* For every $0\le k<p$, if we set

$$\begin{aligned}\Lambda_k(r) &:= \nu_{T\wedge\beta^k}(r)+\int_0^r\left(\frac{t^{2(p-k)}}{r^{2(p-k)}}-1\right)\frac{\nu_{dd^cT\wedge\beta^k}(t)}{t}dt\\ &= r^{2k}\nu_T(r)+\int_0^r\left(\frac{t^{2(p-k)}}{r^{2(p-k)}}-1\right)t^{2k}\frac{\nu_{dd^cT}(t)}{t}dt\\ &=: r^{2k}\nu_T(r)+\mathscr{I}_k(r)\end{aligned}$$

then Λ_k is a non-negative increasing function and $\nu_T(0) = \lim_{r\to 0}\Lambda_0(r)$. Therefore, as in the previous case, one has

$$\begin{aligned} ||\mathcal{T}||(\pi^{-1}(B(\epsilon,r))) &= r^{2p}\nu_T(r) - \epsilon^{2p}\nu_T(\epsilon) + \sum_{k=0}^{p-1} C_p^k(\Lambda_k(r)-\Lambda_k(\epsilon)) \\ &= \sum_{k=0}^{p} C_p^k\left[(r^{2k}\nu_T(r)-\epsilon^{2k}\nu_T(\epsilon)) + (\mathscr{I}_k(r)-\mathscr{I}_k(\epsilon))\right]. \end{aligned}$$

If we tend ϵ to 0 and using the fact that ν_{dd^cT} is a non-positive decreasing function, we obtain

$$\begin{aligned} ||\widehat{T}||(\pi^{-1}(B(r))) &= ||\mathcal{T}||(\pi^{-1}(B(r)\smallsetminus\{0\})) \\ &\leq |\nu_T(r)-\nu_T(0)| + \sum_{k=1}^{p} C_p^k r^{2k}\nu_T(r) + \sum_{k=0}^{p-1} C_p^k \mathscr{I}_k(r) \\ &\leq |\nu_T(r)-\nu_T(0)| + \sum_{k=1}^{p} C_p^k r^{2k}\nu_T(r) \\ &\quad -\sum_{k=1}^{p-1} C_p^k \frac{r^{2k}}{2k}\nu_{dd^cT}(r) + \mathscr{I}_0(r). \end{aligned}$$

which completes the proof.

□

Definition 2. We say that a positive current S satisfies the condition of restriction along an hypersurface Y if for any equation $\{h=0\}$ of Y in a local chart U, one has $\log|h| \in L^1(U,\sigma_S)$ where $\sigma_S = S\wedge\beta^p$ the trace measure associated to S.

The problem now is to give a suitable condition on T to have $\widehat{T}$ satisfies the condition of restriction along the hypersurface $\mathbb{P}^{n-1}$. For this aim, we need the following lemma where we set

$$\Delta := \{z\in\mathbb{C}^n;\ |z_j|<1,\ \forall\, j\in\{1,...,n\}\}$$

the unit polydisc of $\mathbb{C}^n$ and $\Delta^* = \Delta\smallsetminus\{0\}$.

Lemma 5. *Let S be a positive current of bidimension (p,p) on a neighborhood Ω of Δ in $\mathbb{C}^n$. Then the following conditions are equivalent:*

(1) $\log|z_k| \in L^1(\Delta_k^*(1),\sigma_S)$.
(2) $\displaystyle\int_0^1 \frac{\sigma_S(\Delta_k^*(r))}{r}dr < +\infty$, *where* $\Delta_k^*(r) = \{z\in\Delta;\ 0<|z_k|<r\}$ *for all* $1\leq k\leq n$.

This lemma was proved by Raby [7] for a positive closed current. We give the same proof (In fact, only the positivity of the current is needed).

Proof. Equivalence between the two conditions is deduced from the following equality:

$$\int_0^1 \frac{\sigma_S(\Delta_k^*(r))}{r}dr = \int_{\Delta^*} -\log|z_k|d\sigma_S. \tag{3.3}$$

to show (3.3), Let $u \in]0,1]$. Then

$$\int_u^1 \frac{\sigma_S(\Delta_k^*(r))}{r}dr = \int_u^1 \left(\frac{1}{r}\int_{\Delta_k^*(r)} d\sigma_S\right)dr = \int_{D_k(u)} d\sigma_S \otimes \frac{dr}{r}$$

where $D_k(u) = \Delta_k^*(r)\times]u,1[$. Therefore,

$$\begin{aligned}\int_u^1 \frac{\sigma_S(\Delta_k^*(r))}{r}dr &= \int_{\Delta^*}\left(\int_{\max(u,|z_k|)}^1 \frac{dr}{r}\right)d\sigma_S \\ &= \int_{\Delta^*} -\log(\max(u,|z_k|))d\sigma_S\end{aligned}$$

hence,

$$\int_{\Delta^*\smallsetminus\Delta_k^*(u)} -\log|z_k|d\sigma_S \leq \int_u^1 \frac{\sigma_S(\Delta_k^*(r))}{r}dr \leq \int_{\Delta^*} -\log|z_k|d\sigma_S.$$

If we tend u to 0, we obtain equality (3.3). □

Proposition 2. *Let $T \in \mathscr{P}_p^+(\mathbb{C}^n)$ (resp. $T \in \mathscr{P}_p^-(\mathbb{C}^n)$) such that*

$$\int_0^1 \frac{\nu_T(r)-\nu_T(0)}{r}dr < +\infty$$

resp.

$$\int_0^1 \frac{|\nu_T(r)-\nu_T(0)|}{r}dr < +\infty \ \textit{and} \ \int_0^1 \frac{\nu_{dd^cT(r)}}{r}\log r dr < +\infty.$$

Then $\widehat{T}$ satisfies the condition of restriction along $\mathbb{P}^{n-1}$.

This result is due to Giret [5] in the case of positive closed currents.

Proof.

- *First case $T \in \mathscr{P}_p^+(\mathbb{C}^n)$.* Thanks to Inequality (3.1), one has

$$||\widehat{T}||(\pi^{-1}(B(r)) \leq \nu_T(r) - \nu_T(0) + C_r\nu_T(r)$$

  where $C_r = \sum_{k=1}^p C_p^k r^{2k}$. Thanks to Lemma 5, $\widehat{T}$ satisfies the condition of restriction along $\mathbb{P}^{n-1}$ if

$$\int_0^1 \frac{\nu_T(r)-\nu_T(0)}{r}dr < +\infty.$$

- *Second case $T \in \mathscr{P}_p^-(\mathbb{C}^n)$.* As in the previous case, thanks to Inequality (3.2), we have

$$||\widehat{T}||(\pi^{-1}(B(r)) \leq |\nu_T(r) - \nu_T(0)| + C_r\nu_T(r) - C_r'\nu_{dd^cT}(r) + \mathscr{J}_0(r)$$

where $C'_r = \sum_{k=1}^{p} \frac{C_p^k}{2k} r^{2k}$. Thanks to Lemma 5, $\widehat{T}$ satisfies the condition of restriction along $\mathbb{P}^{n-1}$ if

$$\int_0^1 \frac{|\nu_T(r) - \nu_T(0)|}{r} dr < +\infty \quad \text{and} \quad \int_0^1 \mathscr{I}_0(r) dr < +\infty.$$

A simple computation shows that

$$\begin{aligned} \int_0^1 \mathscr{I}_0(r) dr &= \int_0^1 \frac{1}{r} \left(\int_0^r \left(\frac{t^{2p}}{r^{2p}} - 1 \right) \frac{\nu_{dd^cT}(t)}{t} dt \right) dr \\ &= \int_0^1 \frac{\nu_{dd^cT}(t)}{t} \left(\log t - \frac{t^{2p}}{2p} + \frac{1}{2p} \right) dt. \end{aligned}$$

□

As an application of proposition 2, we give a second proof of the Main result.

Corollary 1. *Let T be a positive plurisubharmonic or plurisuperharmonic current as in proposition 2. Then T admits a tangent cone at 0.*

Proof. Let $\mu : \mathbb{C}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$ defined by $\mu(z) = [z]$. Thanks to proposition 2, the current $\mu^\star \left(\widehat{T}_{|\mathbb{P}^{n-1}} \right)$ is positive on $\mathbb{C}^n \smallsetminus \{0\}$ and it admits a trivial extension Θ_T on $\mathbb{C}^n$. We prove that $\Theta_T = \lim_{a \to 0} h_a^\star T$ (see [5]) so Θ_T is the tangent cone to T at 0. □

4. Appendix: Conic currents

Let T be a positive plurisubharmonic or plurisuperharmonic current of bidimension (p,p) on $\mathbb{C}^n$. recall that T is called *conic* if $h_a^\star T = T$ for every $a \in \mathbb{C}^*$. It is well known that $dd^c(h_a^\star T) = h_a^\star(dd^c T)$ so if T is conic then $dd^c T$ is also conic and the two functions ν_T and ν_{dd^cT} are constant. In particular, if $T \in \mathscr{P}_p^{\pm}(\mathbb{C}^n)$ then $\nu_{dd^cT} \equiv \nu_{dd^cT}(0) = 0$ so T is pluriharmonic. The following lemma gives more informations.

Lemma 6. *Let $T \in \mathscr{P}_p^{\pm}(\mathbb{C}^n)$. The following assertions are equivalent:*

1. *T is invariant by dilatations h_a for all $a \in \mathbb{C}^*$;*
2. *T is invariant by dilatations h_a for all $a \in]0, +\infty[$;*
3. *T is pluriharmonic and $T \wedge \alpha^p = 0$ on $\mathbb{C}^n \smallsetminus \{0\}$;*
4. *T is the extension to $\mathbb{C}^n$ of the pull-back of a positive current by the projection $\mu : \mathbb{C}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$.*

Proof. It's clear that (1) implies (2). With the hypothesis of (2) we have $dd^c T$ is also invariant by dilatations h_a for all $a \in]0, +\infty[$ so ν_T and ν_{dd^cT} are constants. Thanks to Lemma 1, one has

$$\int_{B(\epsilon,r)} T \wedge \alpha^p = 0, \quad \forall\, 0 < \epsilon < r.$$

(3) implies (4) and (4) implies (1) are proved by Haggui in [6]. □

Remark 2. The current T_0 of Example 1 is positive plurisuperharmonic conic non pluriharmonic, so if we study positive plurisuperharmonic current non satisfying condition $(C)_0$ then assertion (3) in Lemma 6 may be replaced by $(3)'$ *dd^cT is conic and*

$$\int_{B(\epsilon,r)} T\wedge\alpha^p = \nu_{dd^cT}(0)\log\frac{\epsilon}{r},\quad \forall\, 0<\epsilon<r.$$

Proposition 3. *If $T\in\mathscr{P}_p^+(\mathbb{C}^n)$ (resp. $T\in\mathscr{P}_p^-(\mathbb{C}^n)$) then every adherence value of $(h_a^\star T)_a$ is a positive conic pluriharmonic current on $\mathbb{C}^n$.*

Proof. Let $\Theta=\lim_{k\to+\infty} h_{a_k}^\star T$ where $a_k \underset{k\to+\infty}{\longrightarrow} 0$.

- *First case $T\in\mathscr{P}_p^+(\mathbb{C}^n)$.* Using $h_{a_k}^\star T$ instead of T, the Lelong-Jensen formula gives, for every $0<\epsilon<r$ and k,

$$\begin{aligned}\nu_T(|a_k|r)-\nu_T(|a_k|\epsilon) &= \int_\epsilon^r\left(\frac{1}{t^{2p}}-\frac{1}{r^{2p}}\right)t^{2p-1}\nu_{dd^c(h_{a_k}^\star T)}(t)dt\\ &\quad+\left(\frac{1}{\epsilon^{2p}}-\frac{1}{r^{2p}}\right)\int_0^\epsilon t^{2p-1}\nu_{dd^c(h_{a_k}^\star T)}(t)dt\\ &\quad+\int_{B(\epsilon,r)} h_{a_k}^\star T\wedge\alpha^p.\end{aligned}$$

  If $k\to+\infty$, we obtain

$$\begin{aligned}0 &= \int_{B(\epsilon,r)}\Theta\wedge\alpha^p+\int_\epsilon^r\left(\frac{1}{t^{2p}}-\frac{1}{r^{2p}}\right)t^{2p-1}\nu_{dd^c\Theta}(t)dt\\ &\quad+\left(\frac{1}{\epsilon^{2p}}-\frac{1}{r^{2p}}\right)\int_0^\epsilon t^{2p-1}\nu_{dd^c\Theta}(t)dt.\end{aligned}$$

  Θ is positive plurisubharmonic, hence the three terms of the previous equality are equal to zero. In particular Θ is pluriharmonic and $\Theta\wedge\alpha^p=0$ on $\mathbb{C}^n\smallsetminus\{0\}$. Thanks to Lemma 6, Θ is conic.
- *Second case $T\in\mathscr{P}_p^-(\mathbb{C}^n)$.* Like in the previous case, we consider the non-negative increasing function

$$\Lambda_T(r)=\nu_T(r)+\int_0^r\left(\frac{t^{2p}}{r^{2p}}-1\right)\frac{\nu_{dd^cT}(t)}{t}dt.$$

  We remark that $h_a^\star T\in\mathscr{P}_p^-(\mathbb{C}^n)$ because

$$\int_0^{r_0}\frac{\nu_{dd^c(h_a^\star T)}(t)}{t}dt=\int_0^{|a|r_0}\frac{\nu_{dd^cT}(t)}{t}dt>-\infty,\quad \forall\, a\in\mathbb{C}^*$$

  and

$$\begin{aligned}\Lambda_{h_a^\star T}(r) &= \nu_T(|a|r)+\int_0^r\left(\frac{t^{2p}}{r^{2p}}-1\right)\frac{\nu_{(h_a^\star dd^cT)}(t)}{t}dt\\ &= \nu_T(|a|r)+\int_0^r\left(\frac{t^{2p}}{r^{2p}}-1\right)\frac{\nu_{dd^cT}(|a|t)}{t}dt\\ &= \Lambda_T(|a|r).\end{aligned}$$

Thanks to the proof of theorem 2, for every $0<\epsilon<r$ and k (large enough),

$$\begin{aligned}\Lambda_{h^\star_{a_k}T}(r)-\Lambda_{h^\star_{a_k}T}(\epsilon) &= \Lambda_T(|a_k|r)-\Lambda_T(|a_k|\epsilon)\\ &= \int_{B(\epsilon,r)} h^\star_{a_k}T\wedge\alpha^p\end{aligned}$$

If $k\to+\infty$, we obtain Λ_Θ is constant and $\Theta\wedge\alpha^p=0$ on $\mathbb{C}^n\smallsetminus\{0\}$. So, $\Lambda_\Theta(r)=\nu_\Theta(0)$ for every $r>0$ and this can be written as

$$\nu_\Theta(r)+\int_0^r\left(\frac{t^{2p}}{r^{2p}}-1\right)\frac{\nu_{dd^c\Theta}(t)}{t}dt=\nu_\Theta(0),\quad \forall\, r>0. \tag{4.1}$$

Furthermore, one has

$$\nu_\Theta(r)=\lim_{k\to+\infty}\nu_{h^\star_{a_k}T}(r)=\lim_{k\to+\infty}\nu_T(|a_k|r)=\nu_\Theta(0),\quad \forall\, r>0. \tag{4.2}$$

Equalities (4.1) and (4.2) give

$$\int_0^r\left(\frac{t^{2p}}{r^{2p}}-1\right)\frac{\nu_{dd^c\Theta}(t)}{t}dt=0,\quad \forall\, r>0.$$

Since Θ is a positive plurisuperharmonic current, so $\nu_{dd^c\Theta}$ is non positive, then $\nu_{dd^c\Theta}\equiv 0$. Hence Θ is a positive pluriharmonic current satisfying $\Theta\wedge\alpha^p=0$ on $\mathbb{C}^n\smallsetminus\{0\}$, thanks to lemma 6, Θ is conic. $\square$

Corollary 2. *Let $T\in\mathscr{P}^{\pm}(\mathbb{C}^n)$ and $(a_k)_k,\ (b_k)_k$ are two sequences of complex numbers such that $\left|\frac{a_k}{b_k}\right|$ and $\left|\frac{a_k}{b_k}\right|$ are bounded. If $h^\star_{a_k}T$ and $h^\star_{b_k}T$ converge weakly then $h^\star_{a_k}T-h^\star_{b_k}T$ converges weakly to 0.*

Therefore, the set of adherent values of $(h^\star_aT)_a$ does not change if we restrict to the case $a\in]0,+\infty[$.

Proof. Let $\Theta_1=\lim_{k\to+\infty}h^\star_{a_k}T$ and $\Theta_2=\lim_{k\to+\infty}h^\star_{b_k}T$ such that $c_k=\frac{b_k}{a_k}\underset{k\to+\infty}{\longrightarrow}c\in\mathbb{C}^*$ (we extract subsequences if necessary). For every $\varphi\in\mathscr{D}_{p,p}(\mathbb{C}^n)$, we have

$$\langle h^\star_{a_k}T,\varphi\rangle=\langle h^\star_{1/c_k}h^\star_{b_k}T,\varphi\rangle=\langle h^\star_{b_k}T,h^\star_{c_k}\varphi\rangle\underset{k\to+\infty}{\longrightarrow}\langle\Theta_2,h^\star_c\varphi\rangle$$

hence, $\langle\Theta_1,\varphi\rangle=\langle h^\star_{1/c}\Theta_2,\varphi\rangle=\langle\Theta_2,\varphi\rangle$, because Θ_2 is conic. $\square$

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